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On the homogeneity at infinity of the stationary probability for an affine random walk.

by Y. Guivarc'h, E. Le Page

Abstract

We consider an affine random walk on \mathbb{R} . We assume the existence of a stationary probability ν on \mathbb{R} and we describe the shape at infinity of ν , if ν has unbounded support. We discuss the connections of the result with geometrical or probabilistic problems.

I - Introduction

Let G be the affine group of the line. For $g \in G$, $x \in \mathbb{R}$, we write $gx = a(g)x + b(g)$ with $a(g) \in \mathbb{R}^*$, $b(g) \in \mathbb{R}$. Let μ be a probability on G . We denote by P the Markov operator on \mathbb{R} defined by $P\varphi(x) = \int \varphi(gx)\mu(dg)$ where φ is a bounded Borel function. Our hypothesis H_μ is stated below and we observe that $H_\mu(1)$ and $H_\mu(2)$ imply that P has a unique stationary probability ν (see [16]) ; if $H_\mu(4)$ is also valid, then $\text{supp}\nu$ is unbounded. Here we are interested in the "shape at infinity" of ν ; we will show that for some $\alpha > 0$, the quantities $|t|^\alpha \nu[t, \infty)$ and $|t|^\alpha \nu(-\infty, t]$ have limits at infinity, we discuss their positivity and we illustrate the possible uses of this result by two corollaries in two different contexts. This "homogeneity at infinity" of ν plays an essential role in extreme value theory (see [19]), for random variables associated with the Markov chain X_n^x with kernel P on \mathbb{R} . Also, for random walk in a random medium on \mathbb{Z} (see [21]) the slow diffusion property is closely related to this homogeneity (see [6], [17]). Furthermore the construction of ν given here provides a natural construction of a large class of heavy tailed measures which generates "anomalous" random walks on the additive group \mathbb{R} . This class of measures appears now to be of great interest from the physical point of view (see [2]). In the geometrical context of excursions of geodesic flows on manifolds of negative curvature the "logarithm law" is well known (see [22], [18]), and we will discuss analogous properties for the Markov chain X_n^x .

We assume that μ satisfies the following set of conditions H_μ .

$$H_\mu(1) : \int (|\ell n|a(g)| + |\ell n|b(g)|)\mu(dg) < \infty.$$

$$H_\mu(2) : \text{For some } \alpha > 0 \int |a(g)|^\alpha \mu(dg) = 1.$$

$$H_\mu(3) : \int |a(g)|^\alpha \ell n|a(g)|\mu(dg) < \infty, \int |b(g)|^\alpha \mu(dg) < \infty.$$

$$H_\mu(4) : \text{The elements of } \text{supp}\mu \text{ have no common fixed point in } \mathbb{R}.$$

$$H_\mu(5) : \text{The set } \{\ell n|a(g)| ; g \in \text{supp}\mu\} \text{ generates a dense subgroup of } \mathbb{R}.$$

Then we have the

Theorem 1

Assume that μ satisfies H_μ . Then

1) There exists $c_+ \geq 0$, $c_- \geq 0$ such that $\lim_{t \rightarrow \infty} |t|^\alpha \nu(t, \infty) = c_+$, $\lim_{t \rightarrow -\infty} |t|^\alpha \nu(-\infty, t) = c_-$.

Moreover $c = c_+ + c_- > 0$.

2) If $\mu\{g \in G ; a(g) < 0\} > 0$, then $c_+ = c_- > 0$.

- 3) If $\mu\{g \in G ; a(g) > 0\} = 1$, then $c_+ > 0$ (resp $c_- > 0$) if and only if the action of $\text{supp}\mu$ on \mathbb{R} has no invariant half-line of the form $] - \infty, k]$, (resp $[k, \infty[$) .
- 4) If $\mu\{g \in G ; a(g) < 0\} > 0$, then $\text{supp}\nu = \mathbb{R}$. Otherwise the set $\text{supp}\nu$ is a half-line if and only if $\text{supp}\mu$ preserves a half-line of the same form.

We denote by \mathbb{P} the product probability $\mu^{\otimes \mathbb{N}}$ on $\Omega = G^{\mathbb{N}}$, where \mathbb{N} is the set of positive integers. For $\omega \in \Omega$, we write $g_k = (a(g_k), b(g_k)) = (a_k, b_k)$. Then X_n^x satisfies the stochastic recursion :

$$X_n^x = a_n X_{n-1}^x + b_n, \quad X_0^x = x.$$

The Markov chain X_n^x on \mathbb{R} will be called "affine random walk". It is well known that, for the existence and uniqueness of the stationary measure ν , it is sufficient to assume $H_\mu(1)$

and $\int \ell n|a(g)|\mu(dg) < \infty$; then X_n^x converges in law to $R = \sum_{k=1}^{\infty} a_1 \dots a_{k-1} b_k$ and the law

of R is ν . Also if $\int (|a(g)|^\beta + |b(g)|^\beta)\mu(dg) < \infty$ for some $\beta > 0$, then $\int |x|^\beta d\nu(x) < \infty$. We observe that R can be interpreted as the sum of a "random geometric series", hence its interest for collective risk theory ([19]).

The validity of 1) and 2) was proved in [10], [16] ; in particular implicit expression for c_+, c_- were given in [10] and the relation $c_+ + c_- > 0$ was obtained. Here we restrict our study to 3) and 4), a result which is new under hypothesis H_μ . A different proof was sketched in [12], where a survey of the multidimensional situation was also given. We observe that the main difficulty of the proof occurs when $\text{supp}\mu$ do not preserve a half line and $a(g) > 0$ μ -a.e ; in this case we have $c_+ > 0, c_- > 0$. If $\text{supp}\mu$ has compact support a short complex analytic proof of this fact, depending of a Lemma of E. Landau, well known in analytic number theory, is given in [11] (see also [5]).

We recall that Fréchet's law with parameter γ is the probability Φ_α^γ on \mathbb{R}_+ given by $\Phi_\alpha^\gamma(0, t) = e^{-\gamma t^{-\alpha}}$ where $\gamma > 0, \alpha > 0$. This family of laws is one of the three families of max-infinitely divisible laws of extreme value theory ([9], [19]). The following is shown in [14].

Corollary 2

For $x \in \mathbb{R}$ we denote

$$M_n^x = \sup\{|X_k^x| ; 1 \leq k \leq n\}, \quad {}_+M_n^x = \sup\{X_k^x ; 1 \leq k \leq n\}.$$

Then the sequence $n^{-\alpha} M_n^x$ (resp $n^{-\alpha} {}_+M_n^x$) converges in \mathbb{P} -law to $\Phi_\alpha^{c\theta}$ with $0 < \theta < 1$ (resp $\Phi_\alpha^{c_+\theta_+}$ with $0 < \theta_+ < 1$, if $c_+ > 0$).

Closely related properties have been intensively studied in the context of extreme value theory (see [19]). The positive number θ is the so-called extremal index of the stochastic process X_n^x ; its inverse θ^{-1} gives a measure of the clustering of the exceptionally large values of the process. If the random variables X_n^x were i.i.d. with law ν , one would have $\theta = 1$ (see [9]). If $a(g) > 0, b(g) > 0$ for $g \in \text{supp}\mu$, the above corollary is proved in [15]. It is also known (see [13]) that, under hypothesis H_μ , the normalized Birkhoff sum of X_n^x

converges in law to a stable law of index α if $\alpha < 2$. As mentioned in ([3], remark 4.8), this convergence is a consequence of extreme value properties of X_n^x , at least for $\alpha < 1$. The analysis of random walk in a random medium on \mathbb{Z} developed in [6] is closely related to such properties for the sojourn time of the particle at a site in \mathbb{Z} , instead of its hitting time as in [17], where Birkhoff sums as above played a dominant role. The following logarithm law is an easy consequence of Corollary 2.

Corollary 3

For any $x \in \mathbb{R}$, we have the following $\mathbb{P} - a.e$ convergences :

$$\limsup_{n \rightarrow \infty} \frac{\ell n |X_n^x|}{\ell n(n)} = \frac{1}{\alpha}, \quad \limsup_{n \rightarrow \infty} \frac{\ell n^+(X_n^x)}{\ell n(n)} = \frac{1}{\alpha} \text{ if } c_+ > 0.$$

The so-called "logarithm law" for excursions of geodesic flow around the cusps on hyperbolic manifolds was proved in [22] and extended to more general situations in [18]. It was observed in [20] that in case of the modular surface, it is a simple consequence of Fréchet's law for geodesic flow which follows from already known extreme value properties of the continuous fraction expansion of a number x uniformly distributed on $[0, 1]$ (see [8]).

II - Calculation of invariant measures on \mathbb{R} in a special case

The Lie algebra of G is generated by the vector fields $X = a \frac{\partial}{\partial a}$, $Y = \frac{\partial}{\partial b}$. We consider the convolution semi-group of probability measures on G with infinitesimal generator $D = X^2 + Y^2 - (\beta + 1)X$. This operator is elliptic and we denote by $p^t(t \geq 0)$ the associated semi-group of probability measures.

We have $\int \ln a(g) p^t(dg) = -t(\beta + 1)$ in particular $\int \ln a(g) p(dg)$ is negative if $\beta > -1$, hence p^t has a stationary probability ν on \mathbb{R} in this case. We consider more generally, for any β , the action of p^t and X, Y, D on positive measures of the form $\nu = f(x)dx$ on the line. We denote by X^*, Y^*, D^* the operators adjoint to X, Y, D . Then the extremal solutions of the equation $D^*f = 0$ ($f \geq 0$) are described by the

Proposition 1

With the above notations, the equation $D^*f = 0$ has the following normalized extremal solutions :

$$\begin{aligned} \beta \geq -1 : f(x) &= (1 + x^2)^{-(1+\beta/2)}, \\ \beta < -1 : f_+(x) &= (1 + x^2)^{-(1+\beta/2)} \int_{-\infty}^x (1 + t^2)^{\beta/2} dt, \\ &\text{and } f_-(x) = (1 + x^2)^{-(1+\beta/2)} \int_x^{\infty} (1 + t^2)^{\beta/2} dt. \end{aligned}$$

If $\beta > -1$, then $\int f(x)dx < \infty$. If $\beta \leq -1$ then $\int f(x)dx = \int f_+(x)dx = \int_- f(x)dx = \infty$.

Proof

We calculate the action of X, Y on the measure $\nu = f dx$ as follows.

Since dx is translation-invariant and the action of the one parameter group $x \rightarrow x + b$ is by translation we get $Y^*f = -f'$.

Since $X\varphi(x) = x\varphi'(x)$, we get also $X^*f(x) = -(xf(x))'$. It follows $D^*f(x) = (x(xf(x)))' +$

$f'' + (\beta + 1)(xf)'$, so that the equation $D^*f = 0$ implies :

$$x(xf)' + f'(x) + (\beta + 1)(xf) = k,$$

for a certain constant k , i.e :

$$(1 + x^2)f' + (\beta + 2)(xf) = k.$$

With $u(x) = (1 + x^2)^{-(1+\beta/2)}$ we have $(1 + x^2)u'(x) + (\beta + 2)xu(x) = 0$, hence the above differential equation has the solutions : $f = u(d + kv)$ with $v(x) = \int_0^x (1 + t^2)^{\beta/2} dt$ and d is a constant.

For $\beta \geq -1$, we have $\lim_{x \rightarrow \infty} v(x) = \infty$, hence the condition $f \geq 0$ implies $k = 0$. In this case the equation $D^*f = 0$ has only positive extremal solutions of the form $f(x) = d(1 + x^2)^{-(1+\beta/2)}$. For $\beta = 0$, D is the hyperbolic Laplacian and we recover the Cauchy law on \mathbb{R} with density $\frac{1}{\pi} \frac{1}{1+x^2}$. For $\beta > -1$, we get a probability law with density proportional to $(1 + x^2)^{-(1+\beta/2)}$.

We verify that for $\beta < -1$, the equation $D^*f = 0$ has two basic extremal solutions :

$$f_+(x) = (1 + x^2)^{-(1+\beta/2)} \int_{-\infty}^x (1 + t^2)^{\beta/2} dt,$$

$$f_-(x) = (1 + x^2)^{-(1+\beta/2)} \int_x^{\infty} (1 + t^2)^{\beta/2} dt.$$

The measure ν corresponding to f_+ has infinite mass and satisfies :

$$\lim_{t \rightarrow -\infty} |t|^{2+\beta} \nu(-\infty, t) = c- > 0$$

At $+\infty$ $f_+(x)$ is asymptotic to $c_+ x^{-1}$ with $c_+ > 0$. Analogous properties are valid for f_- . Also, at ∞ , $f(x)$ is asymptotic to $c|x|^{-1}$ ($c > 0$) \square

Remark

The case $\beta > -1$ corresponds to the situation of the theorem with $\alpha = \beta + 1$.

The case $\beta = -1$ corresponds to the (critical) situation of [1], [4]. Then the unique basic extremal solution behaves at infinity like multiplicative Lebesgue measure on \mathbb{R}^* .

The situation $\beta < -1$, with two extremal solutions, corresponds to a so-called phase transition in $P.D.E$ theory, for example in the context of non linear Schrödinger equations.

III - Proof of theorem 1

The proofs of 1) and 2) in [10] are based on the first renewal equation in Lemma 1 below. A delicate point in [10] for the use of the renewal theorem (see [7]) is solved by replacing $^\alpha f(t) = e^{\alpha t} f(t)$ by a related directly Riemann-integrable function. Here we give only the proofs of 3) and 4). We will now assume $\mu\{g ; a(g) > 0\} = 1$ and we will only study the non vanishing of c_+ . To do that we need some preliminary notations and results.

Let T be the stopping time on Ω defined by :

$$T = \{n \geq 1 ; g_1 g_2 \cdots g_n \in G_+\}, \quad T = \infty \text{ if } \{n \geq 1 ; g_1 g_2 \cdots g_n \in G_+\} = \emptyset,$$

where $G_+ = \{b(g) > 0\}$.

We denote by $\bar{\mu}$ the probability on the additive group \mathbb{R} given by $\bar{\mu}(A) = \mu\{\ell n a(g) \in A\}$

Moreover we denote by μ_T the positive measure on \mathbb{R} defined by :

$$\mu_T(A) = \mathbb{P}\{T < +\infty ; \ell n(a_1 a_2 \cdots a_T) \in A\},$$

where A is a Borel subset of \mathbb{R} . We have $\mu_T(\mathbb{R}) = \mathbb{P}(T < +\infty) \leq 1$, and we denote by μ_T^n the n^{th} convolution power of μ_T on the additive group \mathbb{R} . Define f by

$$f(t) = \mathbb{P}\{R > e^t\} = \nu(\cdot]e^t, +\infty[) \quad t \in \mathbb{R},$$

and write $R_n = \sum_{k=1}^n a_1 a_2 \cdots a_{k-1} b_k$, $S_n = \sum_{k=1}^n \ell n(a_k)$.

Then we have the :

Lemma 1

- 1) For every real t , we have $f(t) = \bar{\mu} * f(t) + f_1(t) = \mu_T * f(t) + h_1(t)$ where :
 $f_1(t) = \mathbb{P}\{R - b_1 > e^t\} - \mathbb{P}\{\mathbb{R} > e^t\}$, $h_1(t) = \mathbb{E}\{1_{[T < +\infty]} \nu(\cdot]e^{-S_T}(e^t - R_T), e^{t-S_T})\}$
- 2) For every real t , we have $f(t) = \sum_{n=0}^{+\infty} \mu_T^n * h_1(t) = \sum_{n=0}^{\infty} \mu^n * f_1(t)$.

If p is a bounded measure on \mathbb{R} and φ is a positive Borel function, we write

$$p * \varphi(t) = \int \varphi(t-x)p(dx), \quad t \in \mathbb{R}.$$

We denote by ${}^\alpha\mu$, the probability measure on G defined by : ${}^\alpha\mu(dg) = a^\alpha(g)\mu(dg)$.

We define the probability ${}^\alpha\mathbb{P}$ on $G^{\mathbb{N}}$ by ${}^\alpha\mathbb{P} = {}^\alpha\mu^{\otimes \mathbb{N}}$ and we write ${}^\alpha\mathbb{E}$ for the corresponding expectation.

The measure ${}^\alpha\mu_T$ on \mathbb{R} is defined by ${}^\alpha\mu_T(A) = {}^\alpha\mathbb{E}(1_A(\ell n(a_1 \cdots a_T)))$,

and we write ${}^\alpha h_1(t) = e^{\alpha t} h_1(t) \quad t \in \mathbb{R}$.

Then from lemma 1 we get :

Lemma 2

For every real t we have ${}^\alpha f(t) = \sum_{n=0}^{+\infty} {}^\alpha\mu_T^n * {}^\alpha h_1(t)$

Now we are going to study some properties of T and $\ell n(a_1 a_2 \cdots a_T)$ under ${}^\alpha\mathbb{P}$. For that purpose we consider the new random variables $g'_i (i \geq 1)$ defined by $g'_i = (a_i^{-1}, b_i a_i^{-1})$. Under ${}^\alpha\mathbb{P}$, there random variables are i.i.d with law ${}^\alpha\mu'$. We have :

$$g'_n g'_{n-1} \cdots g'_1 = ((a_1 a_2 \cdots a_n)^{-1}, R_n(a_1 \cdots a_n)^{-1}),$$

hence for $T' = \inf\{n ; g'_n g'_{n-1} \cdots g'_1 \in G_+\}$ we have $T' = T$. It follows that T can be interpreted as the entrance time in $\mathbb{R}_+ =]0, \infty[$ of the affine random walk on \mathbb{R} defined by ${}^\alpha\mu'$, starting from 0. We denote by ${}^\alpha Q$ the Markov kernel of this affine random walk, and for $p \in \mathbb{R}$ we write $p_n = g'_n g'_{n-1} \cdots g'_1 p$.

Lemma 3

1) There exists a unique probability measure ${}^\alpha\nu'$ on \mathbb{R} such that ${}^\alpha Q({}^\alpha\nu') = {}^\alpha\nu'$. The probability ${}^\alpha\nu'$ has no atoms.

2) If ${}^\alpha\nu'(\cdot]0, +\infty] > 0$ then $0 < {}^\alpha\mathbb{E}(T') < \infty$.

Now we complete the proof of Theorem 1 using the above Lemmas.

For assertion 3, there are two cases.

First case ${}^\alpha\nu'(\cdot]0, +\infty] > 0$.

Then by Lemma 3 and the observation before Lemma 3, ${}^\alpha\mathbb{E}(T) = {}^\alpha\mathbb{E}(T') < \infty$, ${}^\alpha\mu_T(\mathbb{R}) = 1$. By Wald's lemma (see [7]), since $T' < \infty$ ${}^\alpha\mathbb{P} - a.e$:

$${}^\alpha\mathbb{E}\{\ell n(a_1 a_2 \cdots a_T)\} = {}^\alpha\mathbb{E}(\ell n(a_1)) {}^\alpha\mathbb{E}(T)$$

where ${}^\alpha\mathbb{E}(\ell n(a_1)) = \mathbb{E}(a_1^\alpha \ell n(a_1))$ is finite and positive, hence ${}^\alpha\mathbb{E}(S_T)$ is finite and positive. Assume $c_+ = 0$, hence $\lim_{t \rightarrow \infty} {}^\alpha f(t) = 0$. Then, if we denote by ${}^\alpha h_{1,L}$ ($L > 0$) the function $t \rightarrow {}^\alpha h_1(t) 1_{[-L,L]}(t)$, we have using Lemma 2 and Proposition A below : for every $L > 0$,

$$0 = \lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t \sum_{n=0}^{+\infty} {}^\alpha \mu_T^n * {}^\alpha h_{1,L}(s) ds = \frac{1}{{}^\alpha\mathbb{E}(\ell n(a_1)) {}^\alpha\mathbb{E}(T)} \int_{-L}^L {}^\alpha h_1(s) ds,$$

hence $0 = \int_{\mathbb{R}} {}^\alpha h_1(s) ds$. Since ${}^\alpha h_1$ and h_1 are non negative we get $h_1 = 0$ $a.e$, hence Lemma 1 implies $f(t) = 0$, $dt - a.e$.

We conclude that for almost every real s :

$$f(s) = \mathbb{P}(R > e^s) = 0,$$

and so $\mathbb{P}(R \leq 0) = 1$, hence $supp \nu \subset]-\infty, 0]$. It follows that $supp \mu$ preserves an interval $(-\infty, v_0]$ with $v_0 \leq 0$.

Second case ${}^\alpha \nu'([0, +\infty]) = 0$.

Denote by $v_0 \leq 0$ the upper bound of the support of the probability ${}^\alpha \nu'$. Then by the stationarity property of ${}^\alpha \nu'$ we can write that for every $n \geq 1$:

$${}^\alpha \mathbb{P}\{g'_n g'_{n-1} \cdots g'_1 v_0 \leq v_0\} = 1, \quad 1 = \mathbb{E}(a_1^\alpha \cdots a_n^\alpha 1_{\{v_0 + R_n \leq a_1 \cdots a_n v_0\}}),$$

which implies that for every integer $n \geq 1$, $\mathbb{P}(R_n \leq -v_0) = 1$ since $\mathbb{E}(a_1^\alpha) = 1$. Since R_n converges $\mathbb{P} - a.e$ to R we have $\mathbb{P}(R \leq -v_0) = 1$ hence $c_+ = 0$.

In conclusion we see that $c_+ = 0$ if and only if the upper bound of $supp \nu$ is finite i.e if $supp \mu$ preserves an interval $] -\infty, -v_0]$.

In order to show assertion 4 we will distinguish the 2 cases $c_+ > 0$, $c_- = 0$, $c_+ > 0$, and $c_- > 0$. We observe that $supp \nu$ is invariant under $supp \mu$ and condition $\int \ell n(a(g)) d\mu(g) < 0$ implies that for some $g \in (supp \mu)^2$ we have $0 < a(g) < 1$. Also the complement of $supp \nu$ is invariant under $(supp \mu)^{-1}$. We denote by T_μ the closed subsemigroup of G generated by $supp \mu$, and by $\Delta \subset \mathbb{R}$ the closure of the set of attractive fixed points of the elements of T_μ . We observe that $T_\mu \Delta \subset \Delta$. Since for any $x \in \Delta$ the law of $g_n \cdots g_1 x$ is supported by Δ and converges to ν , we obtain that $\Delta \supset supp \nu$. Since the attractive fixed points of T_μ belong to $supp \nu$, we conclude that $\Delta = supp \nu$. Then, for any open interval $I = [a, b] \subset \mathbb{R}$, $n < 0$, $g^n(I)$ is an interval of length $a^n(g)(b - a)$ which converges to $+\infty, -\infty$ or \mathbb{R} , depending of the relative positions of I and the fixed point x_0 of g . If $c_+ > 0$ and $c_- = 0$, then from above $supp \mu$ preserves the interval $[\tau, \infty[$ with $\tau = Inf(supp \nu)$. Since $\Delta = supp \nu$ we can choose $g \in (supp \mu)^2$ such that its fixed point $x_0 \in supp \nu$ is arbitrary close to τ , and in particular $\tau \leq x_0 < a$. If $I \subset]\tau, \infty[$ satisfies $\nu(I) = 0$ then $\nu(g^n(I)) = 0$ for $n < 0$; since the length of the interval $g^n(I)$ is $a^n(g)(b - a)$ and $\lim_{n \rightarrow -\infty} a^n(g) = \infty$ this contradicts $c_+ > 0$.

If $c_+ > 0$, $c_- > 0$ the same argument is valid for any interval I with $\nu(I) = 0$. \square

We now give the proofs of the above lemmas.

Proof of Lemma 1

1) Denote $R^n = \sum_{k=n}^{+\infty} a_{n+1} \cdots a_k b_{k+1}$. Under \mathbb{P} the law of R_n is ν and moreover R^n is independant of the random variables $g_i (1 \leq i \leq n)$.

The formula $R = R_n + a_1 \cdots a_n R^n$ gives $R - b_1 = a_1 R^1$, hence :

$$\mathbb{P}\{R - b_1 > e^t\} = \mathbb{P}\{R^1 > e^t a_1^{-1}\} = \mu * f(t), \quad f(t) = \mu * f(t) + f_1(t)$$

We have also from above

$$\begin{aligned} \{R > e^t\} &= \{R_T + a_1 a_2 \cdots a_T R^T > e^t, T < \infty\} \\ &= \{R^T > e^{t-\ell n(a_1 a_2 \cdots a_T)}; T < \infty\} \cup \{e^{t-\ell n(a_1 a_2 \cdots a_T)} < R^T \leq e^{t-\ell n(a_1 a_2 \cdots a_T)}; T < \infty\} \end{aligned}$$

Using the fact that T is a stopping time we have

$$f(t) = \mathbb{P}\{R > e^t\} = \mathbb{P}\{R > e^t, T < \infty\} = \mu_T * f(t) + h_1(t)$$

where $h_1(t) = \mathbb{E}(1_{\{T < \infty\}} \nu[e^{t-\ell n(a_1 \cdots a_T)} - R_T, e^{t-\ell n(a_1 \cdots a_T)}])$,

It follows :

$$f = \sum_{k=0}^n \bar{\mu}^k * f_1 + \bar{\mu}^{n+1} * f$$

where $\bar{\mu}^{n+1} * f(t) = \mathbb{P}\{R > e^t (a_1 \cdots a_{n+1})^{-1}\}$. The condition $\mathbb{E}(\ell n(a_1)) < 0$ implies the $\mathbb{P} - a.e$ convergence of $(a_1 \cdots a_{n+1})^{-1}$ to ∞ , hence $\lim_{n \rightarrow \infty} \bar{\mu}^{n+1} * f(t) = 0$. The first part of the formula follows.

2) From above we deduce that for every integer n and $t \in \mathbb{R}$.

$$f(t) = \sum_{j=0}^n \mu_T^j * h_1(t) + \mu_T^{n+1} * f(t).$$

We now prove that $\lim_{n \rightarrow +\infty} \mu_T^{n+1} * f_1(t) = 0$.

There are two cases

Case 1) $\mathbb{P}(T < \infty) < 1$

We have

$$0 \leq \mu_T^{n+1} * f(t) \leq (\mathbb{P}(T < \infty))^n$$

hence $\lim_{n \rightarrow \infty} \mu_T^{n+1} * f(t) = 0$.

Case 2) $\mathbb{P}(T < \infty) = 1$

Define the shift θ on Ω by $\theta(\omega) = (g_{i+1}(\omega), i \geq 1)$ where $\omega = (g_i(\omega), i \geq 1)$ and consider the sequence $(T_n(\omega))_{n \geq 1}$ of random times defined $\mathbb{P} - a.e$ by $T_{n+1} = T_0 \theta^{T_n}$, $T_1 = T$. Under \mathbb{P} the sequence of random variables $[(T_1, S_{T_1}), \cdots, (T_{n+1} - T_n, S_{T_{n+1}} - S_{T_n})]$, is i.i.d and the law of S_{T_n} is μ_T^n . Because $\mathbb{E}(\ell n(a_1)) < 0$, we have $\mathbb{P} - a.e \lim_{n \rightarrow \infty} S_n = -\infty$ and moreover

$\lim_{n \rightarrow \infty} T_n = \infty$ hence $\mathbb{P} - a.e, \lim_{n \rightarrow \infty} S_{T_n} = -\infty$. We have that

$$\mu_T^{n+1} * f(t) = \mathbb{E}(f(t - S_{T_{n+1}})),$$

and $\lim_{t \rightarrow \infty} f(t) = 0$. So, using Lebesgue's theorem, we can conclude that $\lim_{n \rightarrow \infty} \mu_T^{n+1} * f(t) = 0$

□

Proof of lemma 2

Lemma 2 us a direct consequence of the formula ${}^\alpha \mu_T^n * {}^\alpha h(t) = e^{\alpha t} \mu_T^n * h_1(t)$, Lemma 1 part 2, and the fact that h_1 is non negative. □

Proof of lemma 3

The definition of ${}^\alpha\mu'$ and the condition $H_\mu(3)$ imply $\int |\ell n(a(g))| {}^\alpha\mu'(dg) < \infty$. The strict convexity of the function $\ell n \int a^s(g) \mu(dg) (s > 0)$ gives $\int \ell n(a(g)) {}^\alpha\mu'(dg) < 0$.

It follows $\int |\ell n| b(g) {}^\alpha\mu'(dg) < \infty$.

As observed above, the existence and uniqueness of ${}^\alpha\nu'$ follows.

If x_0 is a fixed point of $\text{supp} {}^\alpha\mu'$ then for any $(a, b) \in \text{supp} \mu$;

$$a^{-1}x_0 + ba^{-1} = x_0, \text{ i.e } x_0(a-1) = b.$$

This implies that $-x_0$ is a fixed point of $\text{supp} \mu$, which contradicts $H_\mu(4)$. Hence, as it well known (see [5]), ${}^\alpha\nu'$ has no atom.

In order to show ${}^\alpha\mathbb{E}(T') < \infty$ we consider the space ${}^a\Omega^\# = \mathbb{R} \times G^\mathbb{Z}$ and the extended bilateral shift defined by ${}^a\theta(p, \omega) = (p_1, \theta\omega)$ where $p_1 = g'_1(p)$ and θ is the bilateral shift on $G^\mathbb{Z}$. We endow ${}^a\Omega^\#$ with the Markov measure $\kappa^\#$ associated with the ${}^\alpha Q$ -invariant probability ${}^\alpha\nu'$. Clearly $\kappa^\#$ is ${}^a\theta$ -invariant and ergodic. Also we consider the fibered bilateral Markov chain (p_n, V_n) on $\mathbb{R} \times \mathbb{R}^*$ where $V_n = p^{-1}p_n(a_1a_2 \cdots a_n) = p^{-1}(p + R_n)$. Let τ be the first "ladder epoch" of (p_n, V_n) (see [7]), i.e $\tau = \text{Inf}\{n \geq 1 ; V_n > 1\}$, hence $p^{-1}p_\tau > 0$ and $\tau = T$ if $p > 0$. We observe that the conditions in $H_\mu(3)$ implies $\int |p|^\varepsilon {}^\alpha\nu'(dp) < \infty$ for some $\varepsilon > 0$, hence $\limsup_{|n| \rightarrow \infty} \frac{1}{|n|} \ell n |p_n| \leq 0$. Since ${}^\alpha\mathbb{E}(\ell n(a_1)) > 0$ the

ergodic theorem gives $\lim_{n \rightarrow \infty} |V_n| = \infty$, $\lim_{n \rightarrow -\infty} |V_n| = 0$, in particular τ is finite $\kappa^\#$ -a.e.

Since ${}^\alpha\nu'(\mathbb{R}_+) > 0$ we can consider the Markov kernel ${}^\alpha Q_+$ induced by ${}^\alpha Q$ on \mathbb{R}_+ ; the normalized restriction ${}^\alpha\nu'_+$ of ${}^\alpha\nu'$ to \mathbb{R}_+ is ${}^\alpha Q_+$ -invariant and ergodic. We denote by ${}^a\Omega_+^\#$ the subset of ${}^a\Omega^\#$ defined by the conditions $p_n > 0$ infinitely often for $n = n_k > 0$ and $n = n_{-k} < 0$. Since ${}^\alpha\nu'(\mathbb{R}_+) > 0$, ${}^a\Omega_+^\#$ has positive $\kappa^\#$ -measure and we denote by $\kappa_+^\#$ the normalized restriction of $\kappa^\#$ to ${}^a\Omega_+^\#$; then $\kappa_+^\#$ is invariant and ergodic under the corresponding induced shift ${}^a\theta_+$. From above we know that $\lim_{k \rightarrow \infty} V_{n_{-k}} = 0$, hence the time

$\tau_+(\omega) = n_{-j}$ ($j \geq 0$), of the last strict maximum of $V_{n_{-k}}$ is finite $\kappa_+^\#$ -a.e. We define ${}^a\Omega_0^\# = \{\sup_{k > 0} V_{n_{-k}} < 1\} = \{\tau_+ = 0\}$. Then we have ${}^a\kappa_+^\#({}^a\Omega_0^\#) > 0$ since, by ${}^a\theta_+$ -invariance

of $\kappa_+^\#$:

$$\begin{aligned} 1 &= \kappa_+^\#\{\tau_+ > -\infty\} = \sum_{n \geq 0} \kappa_+^\#\{\tau_+ = -n\} \leq \sum_{n \geq 0} \kappa_+^\#\{V_{-n} > \sup_{n_{-k} < -n} V_{n_{-k}}\} = \\ &\sum_{n \geq 0} \kappa_+^\#\{0 > \sup_{k > 0} V_{n_{-k}}\} \leq \infty \kappa_+^\#({}^a\Omega_0^\#). \end{aligned}$$

On the other hand, the definition of τ shows that for $\omega \in {}^a\Omega_0^\#$, $\tau(\omega)$ is the first return time of ${}^a\theta^k(\omega)$ to ${}^a\Omega_0^\#$, so that ${}^a\theta^\tau$ is the transformation of ${}^a\Omega_0^\#$ induced by ${}^a\theta$ on ${}^a\Omega_0^\#$. Then Kac's theorem (see [23]) implies that ${}^a\theta^\tau$ is ergodic with respect to the normalized restriction $\kappa_0^\#$ of $\kappa_+^\#$ to ${}^a\Omega_0^\#$ and ${}^\alpha\mathbb{E}(\tau) = \int \tau(\omega) \kappa_0^\#(d\omega) < \infty$. Also we denote by ${}^\alpha\nu_+^\tau$ the push forward of $\kappa_0^\#$ to \mathbb{R}_+ under the map $\omega \rightarrow p_0(\omega)$. Since the stopped kernel ${}^\alpha Q_+^\tau$ and the map τ commute with p_0 , the measure κ_+^τ is ${}^\alpha Q_+^\tau$ -invariant, ergodic and absolutely continuous with respect to ${}^\alpha\nu_+$ with ${}^\alpha\mathbb{E}_0(\tau) = {}^\alpha\mathbb{E}(T) < \infty$. \square

Remark

A different proof of ${}^\alpha\mathbb{E}(T) < \infty$ uses the interpretation of $T = T'$ as hitting time of the open set \mathbb{R}_+ by the Markov chain with kernel ${}^\alpha Q$ starting from 0. Since $\int a^\delta(g) {}^\alpha\mu'(dg) < \infty$, $\int |b^\delta(g)| {}^\alpha\mu'(dg) < \infty$ for $0 < \delta < \alpha$, the operator defined by ${}^\alpha Q$ on a space of Hölder functions on \mathbb{R} (as in [13]) has a spectral gap. This implies ${}^\alpha\mathbb{E}(T') < \infty$. The proof given above extends to the multidimensional case.

IV - Appendix : a weak renewal theorem

Proposition A Let $(Z_n)_{n \geq 1}$ a sequence of independant, identically distributed real random variables on \mathbb{R} with law η . Assume that $\int |z|\eta(dz) < +\infty$ and that $\gamma = \int z\eta(dz) > 0$. Let ψ a bounded non negative Borel function which is supported on $[-a, a]$.

Then the potential $U\psi = \sum_{n=0}^{+\infty} \eta^n * \psi$ is a bounded function and we have

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t U\psi(s) ds = \frac{1}{\gamma} \int_{\mathbb{R}} \psi(t) dt$$

Proof

If $\Sigma_n = \sum_{i=1}^n Z_i$, we have : $U\psi(s) = \sum_{n=0}^{+\infty} E[\psi(s - \Sigma_n)]$.

Because $\gamma = \int z\eta(dz) > 0$ the random walk on \mathbb{R} with law η is transient and using the maximum principle we have that $\sup_{s \in \mathbb{R}} U\psi(s) < +\infty$.

For $\varepsilon > 0$, $t > 0$ denote

$$\begin{aligned} n_1(t) &= [\frac{1}{\gamma}\varepsilon t] = n_1, & n_2(t) &= [\frac{1}{\gamma}(1 - \varepsilon)t] = n_2, \\ U_n\psi &= \sum_0^{n-1} \eta^k * \psi, & U_n\psi &= \sum_{n+1}^{\infty} \eta^k * \psi, & U_n^m\psi &= \sum_n^m \eta^k * \psi. \end{aligned}$$

Then we have

$$I(t) = \frac{1}{t} \int_0^t U\psi(s) ds = \sum_1^3 I_k(t) - I_4(t)$$

where

$$\begin{aligned} I_1(t) &= \frac{1}{t} \int_{\mathbb{R}} U_{n_1}^{n_2} \psi(s) ds, & I_2(t) &= \frac{1}{t} \int_{\mathbb{R}} U_{n_1} \psi(s) ds, \\ I_3(t) &= \frac{1}{t} \int_{\mathbb{R}} U^{n_2} \psi(s) ds, & I_4(t) &= \frac{1}{t} \int_{\mathbb{R}[0, t]} U_{n_1}^{n_2} \psi(s) ds. \end{aligned}$$

We have

$$\begin{aligned} I_1(t) &= \frac{n_2 - n_1 + 1}{t} (\int_{\mathbb{R}} \psi(s) ds) \text{ hence } \lim_{t \rightarrow +\infty} I_1(t) = \frac{(12\varepsilon)}{\gamma} \int_{\mathbb{R}} \psi(s) ds, \\ 0 \leq I_4(t) &\leq (\frac{(n_2 - n_1 + 1)}{t} \sup_{s \in \mathbb{R}} |\psi(s)|) \sup_{n_1 \leq n \leq n_2} [\mathbb{P}(\Sigma_n \leq a) + \mathbb{P}(t - \Sigma_n \leq a)]. \end{aligned}$$

By the law of large numbers we know that $\mathbb{P} - a.e$, $\lim_{n \rightarrow +\infty} \frac{\Sigma_n}{n} = \gamma > 0$, hence :

$$\lim_{t \rightarrow +\infty} \sup_{n_1 \leq n \leq n_2} (\mathbb{P}\{\Sigma_n \leq a\} + \mathbb{P}\{t - \Sigma_n \leq a\}) = 0.$$

Hence $\lim_{t \rightarrow +\infty} I_4(t) = 0$ and : $0 \leq \frac{I_2(t)}{t} \leq \frac{\varepsilon}{\gamma} \times \int \psi(s) ds$.

Consider now $I_3(t)$ and denote for $n \in \mathbb{N}$, $s > 0$: $\rho_n^s = \text{Inf}\{k \geq n ; |V_n - s| \leq a\}$.

We use the interpretation of $U^n\psi$ as the expected number of visits to ψ after time n : $U^n\psi(x) \leq (U\psi)\mathbb{P}\{\rho_n^s < \infty\}$ with $n\frac{[(1+\varepsilon)t]}{\gamma} = n_2$, hence :

$$I_3(t) \leq |U\psi|\mathbb{P}\{\Sigma_k \leq t + a \text{ for some } k \geq \frac{(1+\varepsilon)t}{\gamma}\}.$$

Since $\frac{\Sigma_n}{n}$ converges to γ , $\mathbb{P} - a.e$, we get $\lim_{t \rightarrow \infty} I_3(t) = 0$.

Since ε is arbitrary we get finally : $\lim_{t \rightarrow \infty} I(t) = \frac{1}{\gamma} \int \psi(s)ds$. \square

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